

CHAPTER 5: CAPACITORS AND INDUCTORS

5.1 Introduction

- Unlike resistors, which dissipate energy, capacitors and inductors store energy.
- Thus, these passive elements are called *storage* elements.

5.2 Capacitors

- Capacitor stores energy in its electric field.
- A capacitor is typically constructed as shown in Figure 5.1.

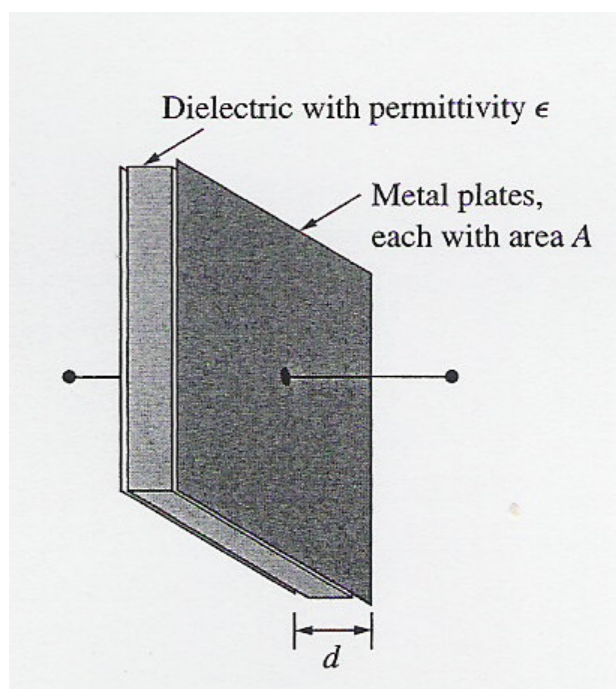


Figure 5.1

A **capacitor** consists of two conducting plates separated by an insulator (or dielectric)

- When a voltage v is applied, the source deposits a positive charge q on one plate and negative charge $-q$ on the other.

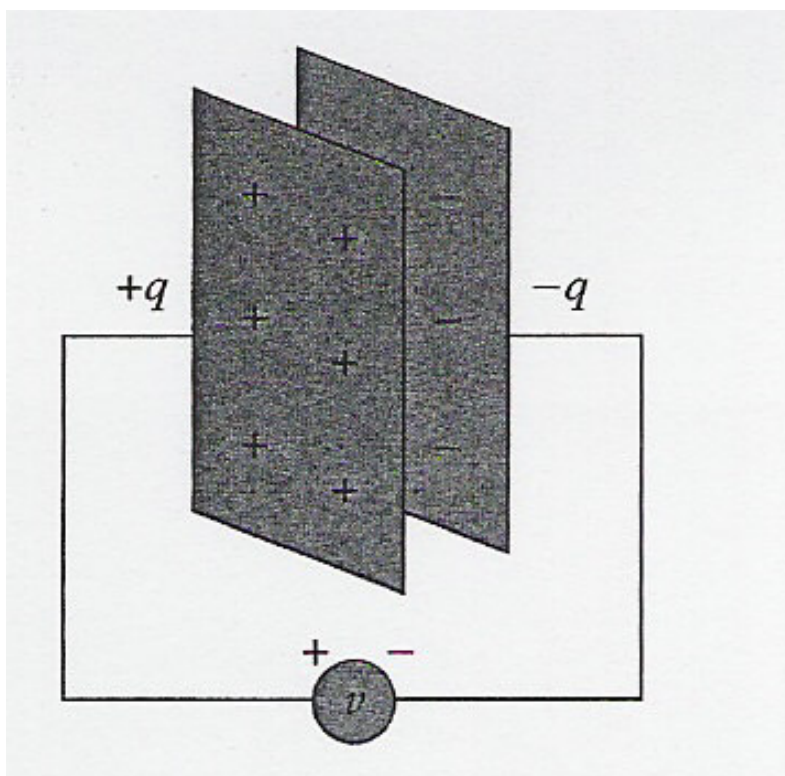


Figure 5.2

- The charge stored is proportional to the applied voltage, v

$$q = Cv \quad (5.1)$$

where C is the constant of proportionality, which is known as the *capacitance* of the capacitor.

- Unit for capacitance: farad (F).

- Definition of capacitance:

Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates.

- Capacitance is depends on the physical dimensions of the capacitor.
- For parallel-plate capacitor, capacitance is given by

$$C = \frac{\epsilon A}{d} \quad (5.2)$$

where A is the surface area of each plate

d is the distance between the plates

ϵ is the permittivity of the dielectric material between the plates

- The symbol of capacitor:

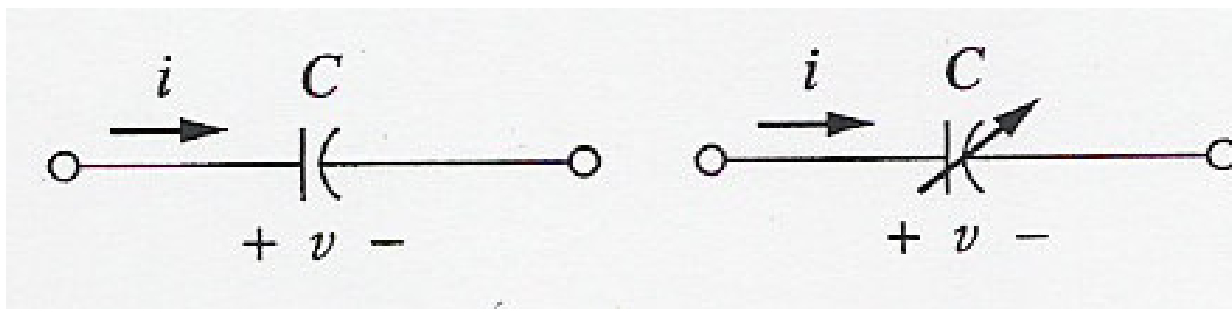


Figure 5.3

- The current flows into the positive terminal when the capacitor is being charged.
- The current flows out of the positive terminal when the capacitors is discharging.
- Differentiating both sides of Equation 5.1,

$$\frac{dq}{dt} = C \frac{dv}{dt}$$

Thus,

$$i = C \frac{dv}{dt} \quad (5.3)$$

- Capacitors that satisfy Equation 5.3 are said to be linear.
- The voltage-current relation:

$$v = \frac{1}{C} \int_{-\infty}^t i(t) dt$$

$$v = \frac{1}{C} \int_{t_0}^t i(t) dt + v(t_0) \quad (5.4)$$

where $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0 .

- Thus, the capacitor voltage is depends on the past history of the capacitor current – has memory.

- The instantaneous power given by:

$$p = vi = Cv \frac{dq}{dt} \quad (5.5)$$

- The energy stored given by:

$$w = \int_{-\infty}^t p dt = C \int_{-\infty}^t v \frac{dv}{dt} dt = C \int_{-\infty}^t v dv = \frac{1}{2} Cv^2 \Big|_{t=-\infty}^t$$

Note that $v(-\infty) = 0$ because the capacitor was uncharged at $t = -\infty$.

Thus,

$$w = \frac{1}{2} Cv^2 = \frac{q^2}{2C} \quad (5.6)$$

- Four issues:

- (i) From Equation 5.3, when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero.

A capacitor is an open circuit to dc.

- (ii) The voltage on the capacitor must be continuous. The capacitor resists an abrupt change in the voltage across it. According to

Equation 5.3, discontinuous change in voltage requires an infinite current, which is physically impossible.

- (iii) The ideal capacitor does not dissipate energy.
- (iv) A real, nonideal capacitor has a parallel-model leakage resistance.

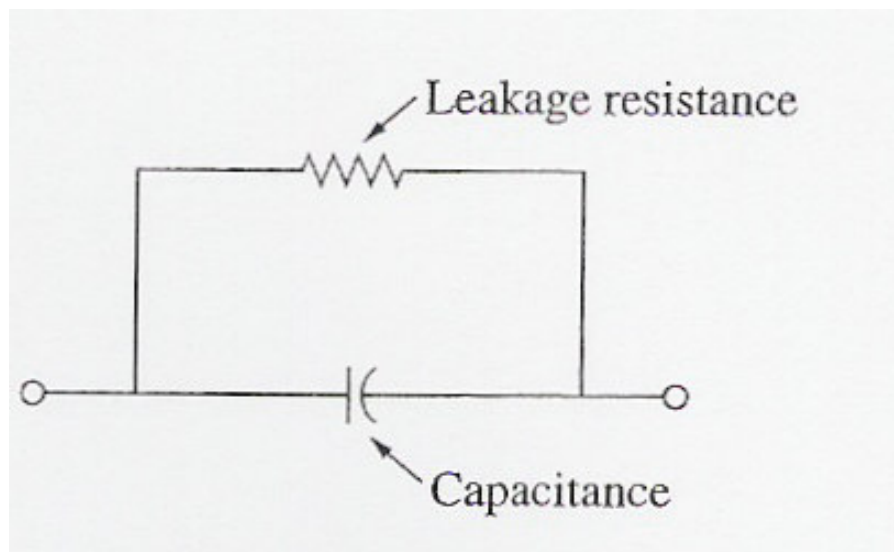


Figure 5.4

- Example 1:

The voltage across a $5\mu\text{F}$ capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

$$i(t) = C \frac{dv}{dt} = 5 \times 10^{-6} \frac{d}{dt} (10 \cos 6000t)$$

$$i(t) = -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t$$

$$= -0.3 \sin 6000t$$

- Example 2:

An initially charged 1-mF capacitor has the current as shown in Figure 5.5. Calculate the voltage across it at $t = 2$ ms and $t = 5$ ms.

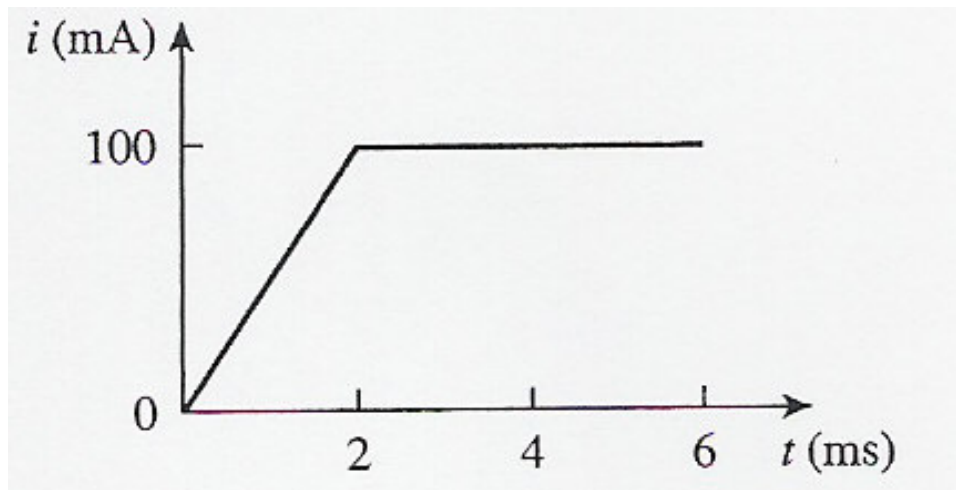


Figure 5.5

The current waveform can be described mathematically as:

$$i(t) = \begin{cases} 50t \text{ mA} & 0 < t < 2 \\ 100 \text{ mA} & t > 2 \end{cases}$$

For $t = 2$ ms:

$$v = \frac{1}{C} \int_{t_0}^t i(t) dt + v(t_0) = \frac{1}{10^{-3}} \int_0^2 50 \times 10^{-3} t + v(0)$$

$$v = \frac{1}{10^{-3}} \left. \frac{50 \times 10^{-3} t^2}{2} \right|_0^2 + 0 = 100 \text{ mV}$$

For $t = 5$ ms:

$$v = \frac{1}{C} \int_{t_0}^t i(t) dt + v(t_0)$$

$$= \frac{1}{10^{-3}} \left(\int_0^2 50 \times 10^{-3} t + \int_2^5 100 \times 10^{-3} \right) + v(0)$$

$$v = \frac{1}{10^{-3}} \left(\left. \frac{50 \times 10^{-3} t^2}{2} \right|_0^2 + 100t \Big|_2^5 \right) + 0 = 400 \text{ mV}$$

5.3 Series and Parallel Capacitors

- Consider the circuit as shown in Figure 5.6 with N capacitors in parallel:

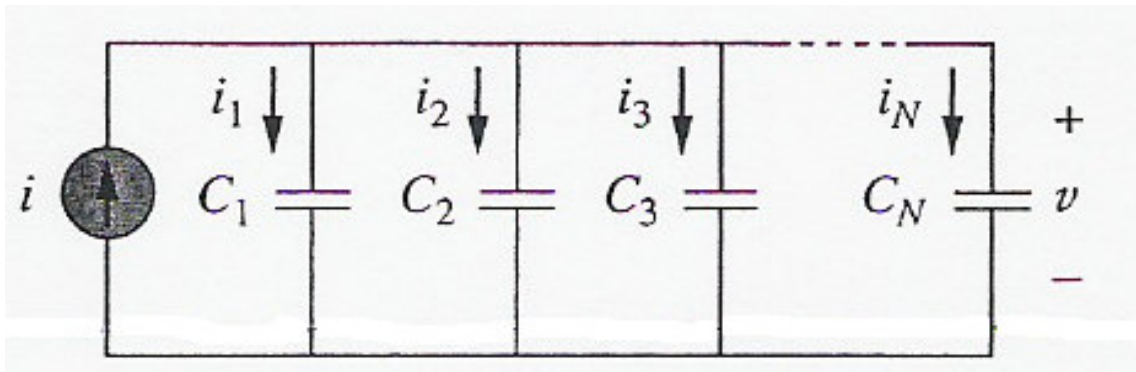


Figure 5.6

Applying KCL:

$$i = i_1 + i_2 + \dots + i_N$$

But,

$$i_k = C_k \frac{dv}{dt}$$

Hence,

$$\begin{aligned} i &= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \dots + C_N \frac{dv}{dt} \\ &= \left(\sum_{k=1}^N C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt} \end{aligned}$$

where,

$$C_{eq} = C_1 + C_2 + \dots + C_N \quad (5.7)$$

- Consider the circuit as shown in Figure 5.7 with N capacitors in series:

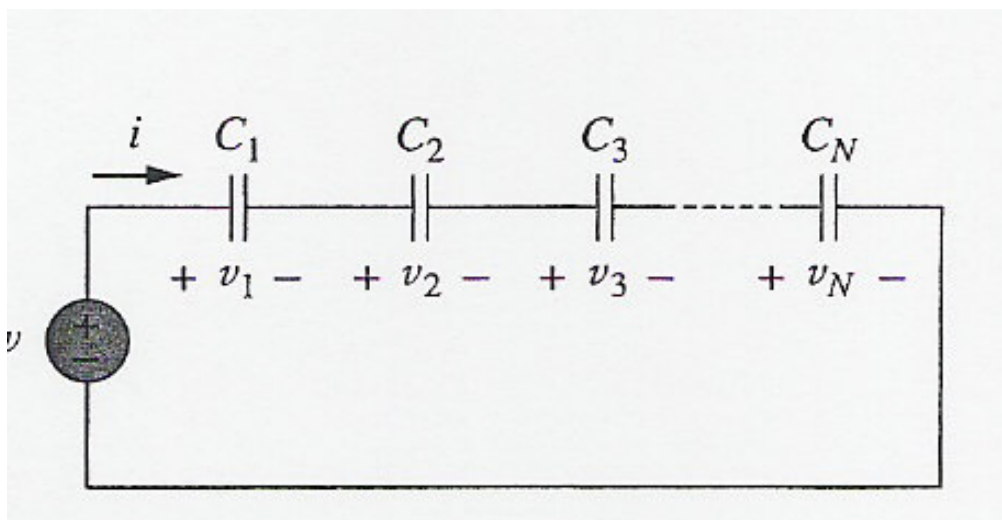


Figure 5.7

Applying KVL,

$$v = v_1 + v_2 + \dots + v_N$$

But,

$$v_k = \frac{1}{C_k} \int_{t_0}^t i(t) dt + v_k(t_0)$$

Thus,

$$v = \frac{1}{C_1} \int_{t_0}^t i(t) dt + v_1(t_0) + \frac{1}{C_2} \int_{t_0}^t i(t) dt + v_2(t_0) \\ + \dots + \frac{1}{C_N} \int_{t_0}^t i(t) dt + v_N(t_0)$$

$$v = \left(\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_N} \right) \int_{t_0}^t i(t) dt$$

$$+ v_1(t_0) + v_2(t_0) + \dots + v_N(t_0)$$

$$v = \frac{1}{C_{eq}} \int_{t_0}^t i(t) dt + v(t_0)$$

where,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_N} \quad (5.8)$$

- Example:

Find the equivalent capacitance seen at terminals of the circuit in Figure 5.8.

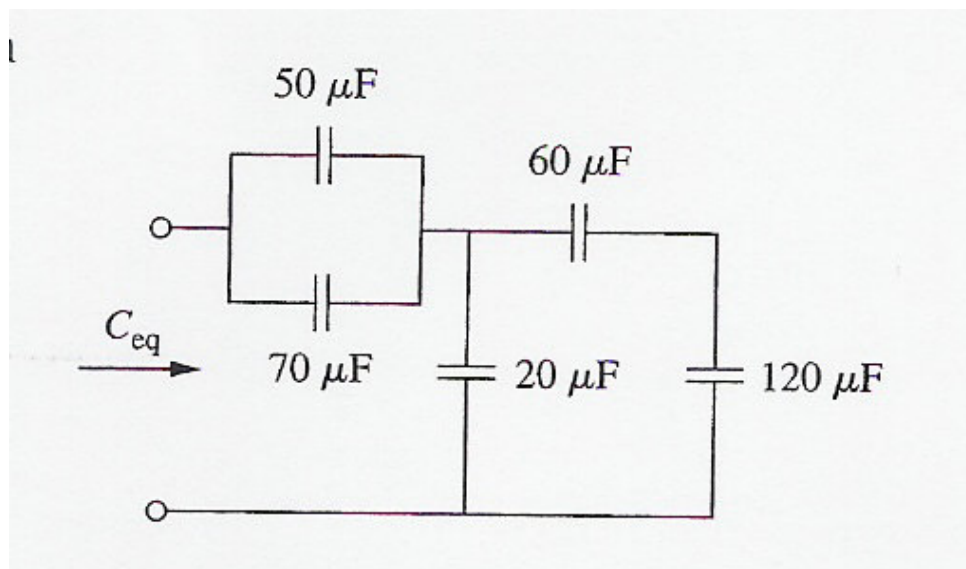


Figure 5.8

$$C_{eq} = (50\mu + 70\mu) \parallel (20\mu + (60\mu \parallel 120\mu))$$

$$= 40\mu F$$

- Example 2:

Find the voltage across each of the capacitors in Figure 5.9.

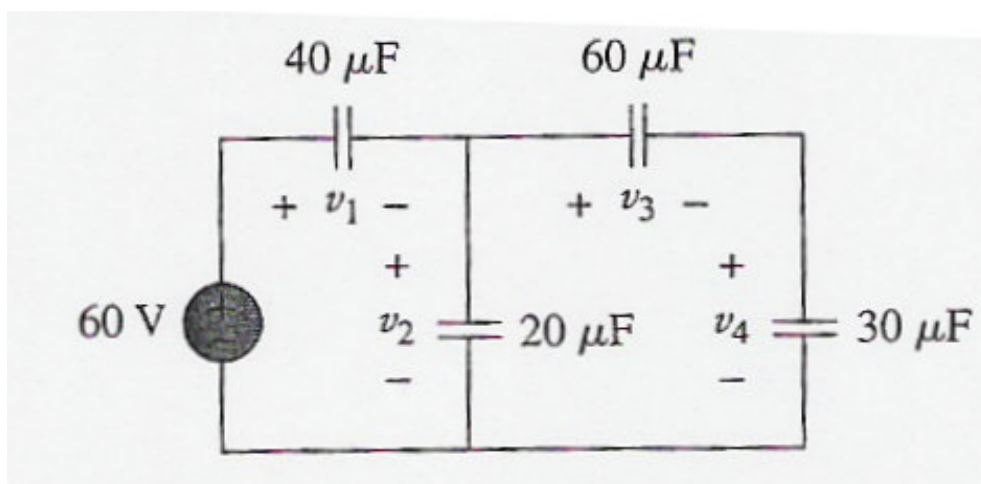


Figure 5.9

$$C_{eq} = (40\mu) \parallel (20\mu + (60\mu \parallel 30\mu))$$

$$C_{eq} = 20\mu F$$

The total charge,

$$q = C_{eq} v = 20 \times 10^{-6} \times 60 = 1.2 \times 10^{-3} \text{ C}$$

$$\therefore v_1 = \frac{q}{C_1} = \frac{1.2 \times 10^{-3}}{40 \times 10^{-6}} = 30 \text{ V}$$

Using KVL,

$$60 \text{ V} = v_1 + v_2$$

$$\therefore v_2 = 30 \text{ V}$$

For Loop 2:

$$v_2 = v_3 + v_4$$

But,

$$q_{Loop2} = C_{60\mu\parallel 30\mu} v_2$$

$$q_{Loop2} = (20 \times 10^{-6})(30) = 0.6 \times 10^{-3} \text{ C}$$

$$v_3 = \frac{q_{Loop2}}{C_3} = \frac{0.6 \times 10^{-3}}{60 \times 10^{-6}} = 10 \text{ V}$$

$$v_4 = v_2 - v_3 = 20 \text{ V}$$

5.4 Inductors

- Inductor is a passive element designed to store energy in its magnetic field.
- Any conductor of electric current has inductive properties and may be regarded as an inductor.
- To enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire.

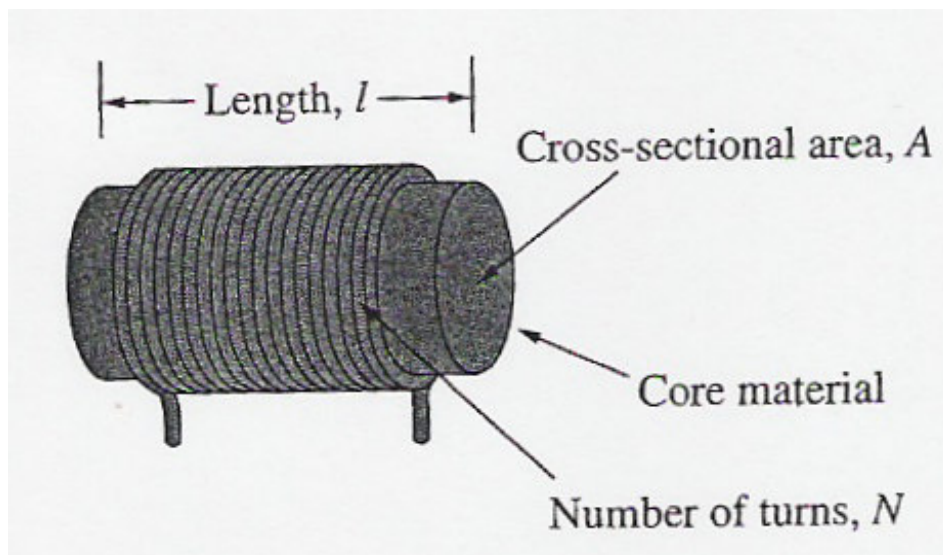


Figure 5.10

An **inductor** consists of a coil of conducting wire.

- If the current passes through an inductor, the voltage across the inductor is proportional to the time of change of the current.

- In mathematical form:

$$v = L \frac{di}{dt} \quad (5.9)$$

where L is the constant of proportionality called the inductance of the inductor.

- The unit of inductance is *henry (H)*.

Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it

- The inductance depends on inductor's physical dimension and construction, which is given by:

$$L = \frac{N^2 \mu A}{l} \quad (5.10)$$

where N is the number of turns

l is the length

A is the cross sectional area

μ is the permeability of the core

- The symbol of inductor:

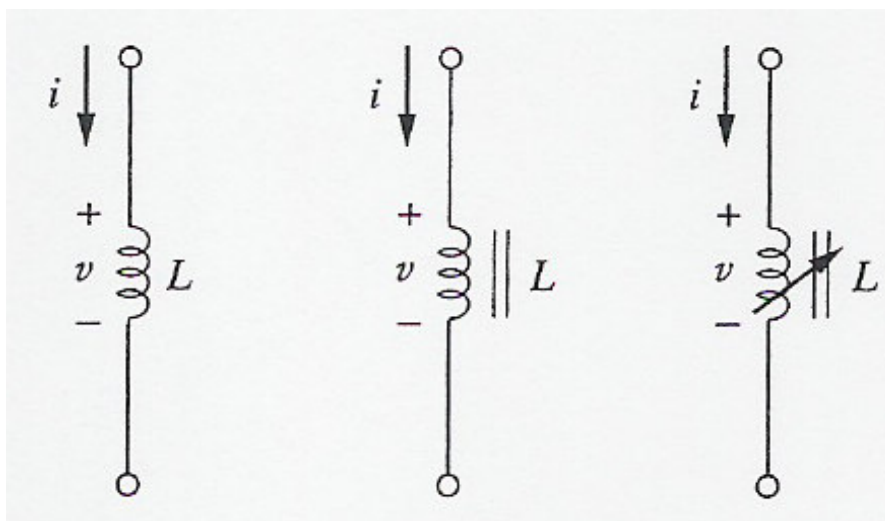


Figure 5.11

- The current-voltage relationship:

$$di = \frac{1}{L} v dt$$

Integrating gives,

$$i = \frac{1}{L} \int_{-\infty}^t v(t) dt$$

or

$$i = \frac{1}{L} \int_{t_o}^t v(t) dt + i(t_o) \quad (5.11)$$

where $i(t_o)$ is the total current for $-\infty < t < t_o$

$i(-\infty) = 0$. (Note: there must be a time in the past when there was no current in the inductor.)

- The inductor stores energy in its magnetic field.
- The power delivered to the inductor:

$$p = iv = \left(L \frac{di}{dt} \right) i \quad (5.12)$$

- The energy stored:

$$w = \int_{-\infty}^t p dt = \int_{-\infty}^t \left(L \frac{di}{dt} \right) i dt$$

$$w = L \int_{-\infty}^t i di = \frac{1}{2} Li^2(t) - \frac{1}{2} Li^2(-\infty)$$

since $i(-\infty) = 0$,

$$w = \frac{1}{2} Li^2 \quad (5.13)$$

- 4 issues:
 - (i) From equation 5.9, the voltage across an inductor is zero when the current is constant (i.e. dc source).

An inductor acts like a short circuit to dc.

- (ii) An important property of the inductor is its opposition to the change in current flowing through it.

The current through an inductor cannot change instantaneously.

According to Equation 5.9, discontinuous change in current requires an infinite voltage, which is physically impossible.

- (iii) The ideal inductor does not dissipate energy.
- (iv) A real, nonideal inductor has a serial-model resistance. This resistance is called a winding resistance, R_w .

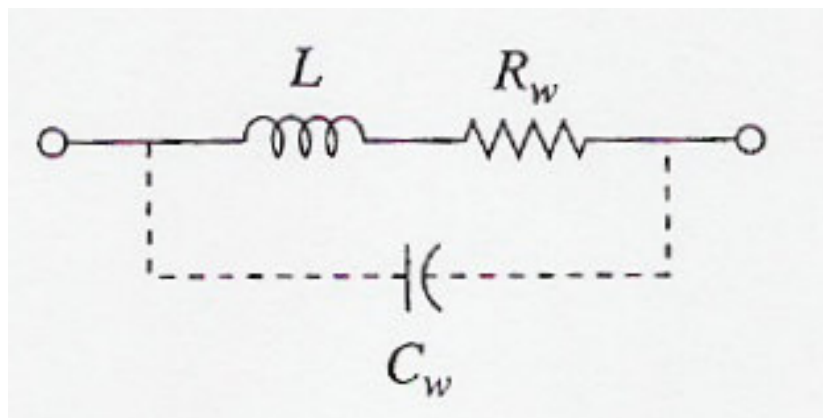


Figure 5.12

- Example 1:

If the current through a 1 mH inductor is $i(t) = 20 \cos 100t$ mA, find the terminal voltage and the energy stored.

The terminal voltage,

$$v = L \frac{di}{dt} = 10^{-3} \times 20 \times 100 \times -\sin 100t$$

$$v = -2 \sin 100t \text{ mV}$$

The energy stored,

$$w = \frac{1}{2} Li^2 = \frac{1}{2} (10^{-3}) (20 \times 10^{-3} \cos 100t)^2$$

$$\therefore w = 0.2 \cos^2 100t \mu\text{J}$$

- Example 2:

Consider the circuit as shown in Figure 5.13. under dc conditions, find (a) i , v_c and i_L , (b) the energy stored in the capacitor and inductor.

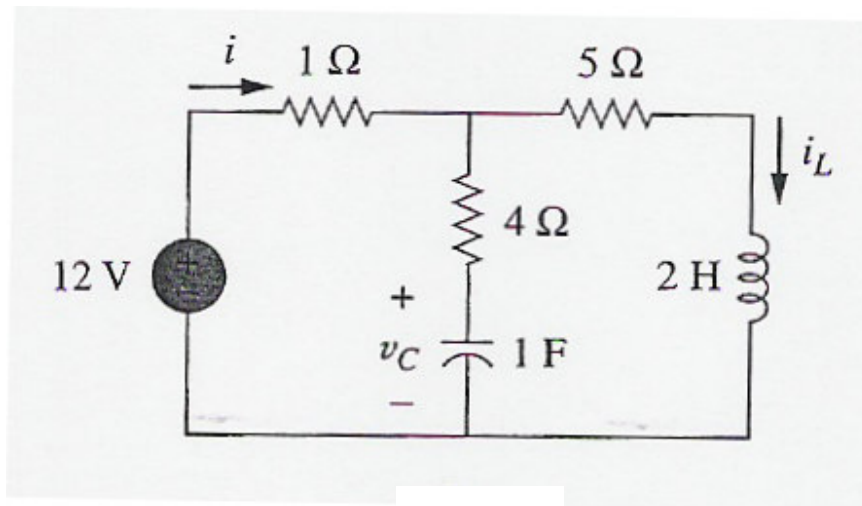


Figure 5.13

- (a) Under dc condition;
 The capacitor – open circuit
 The inductor – short circuit

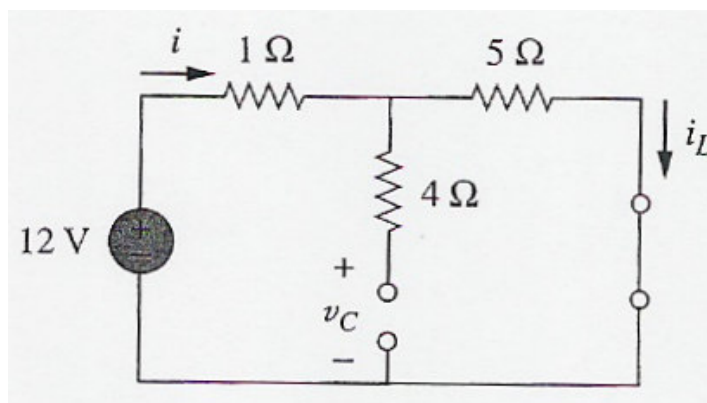


Figure 5.14

From Figure 5.14,

$$i = i_L = \frac{12}{1+5} = 2 \text{ A}$$

$$v_C = i_L R_{5\Omega} = (2)(5) = 10 \text{ V}$$

(b) The energy in the capacitor,

$$w_C = \frac{1}{2} C v_C^2 = \frac{1}{2} (1)(10^2) = 50 \text{ J}$$

The energy in the inductor,

$$w_L = \frac{1}{2} L i_L^2 = \frac{1}{2} (2)(2^2) = 4 \text{ J}$$

5.5 Series and Parallel Inductors

- Consider the circuit as shown in Figure 5.15 with N inductors in series:

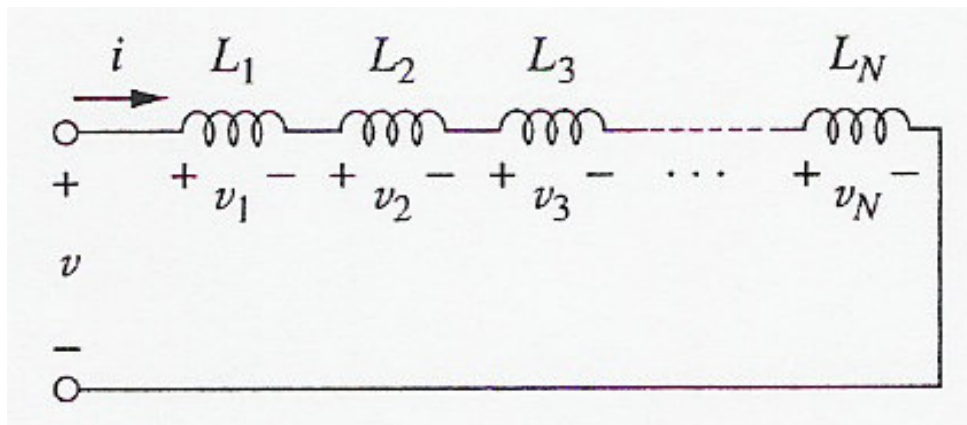


Figure 5.15

Applying KVL:

$$v = v_1 + v_2 + \dots + v_N$$

But,

$$v_k = L_k \frac{di}{dt}$$

Hence,

$$\begin{aligned} v &= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \dots + L_N \frac{di}{dt} \\ &= \left(\sum_{k=1}^N L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt} \end{aligned}$$

where,

$$L_{eq} = L_1 + L_2 + \dots + L_N \quad (5.14)$$

- Consider the circuit as shown in Figure 5.16 with N inductors in parallel:

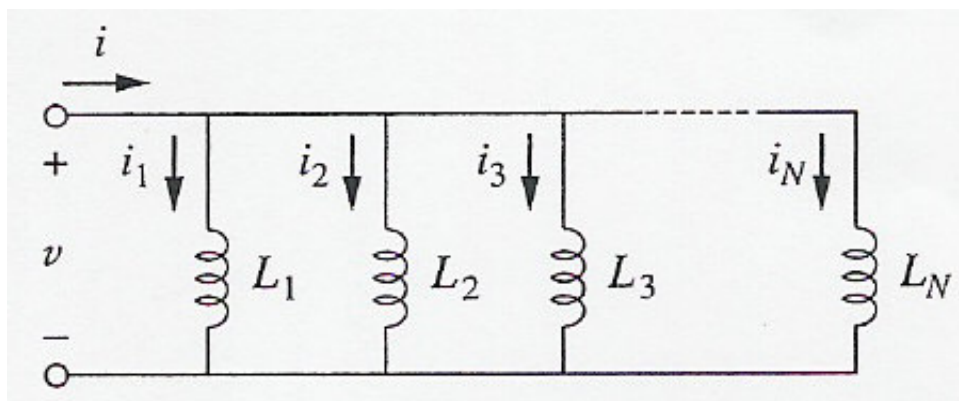


Figure 5.16

Applying KCL,

$$i = i_1 + i_2 + \dots + i_N$$

But,

$$i_k = \frac{1}{L_k} \int_{t_0}^t v(t) dt + i_k(t_0)$$

Thus,

$$\begin{aligned} i &= \frac{1}{L_1} \int_{t_0}^t v dt + i_1(t_0) + \frac{1}{L_2} \int_{t_0}^t v dt + i_2(t_0) \\ &\quad + \dots + \frac{1}{L_N} \int_{t_0}^t v dt + i_N(t_0) \\ i &= \left(\frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_N} \right) \int_{t_0}^t v dt \\ &\quad + i_1(t_0) + i_2(t_0) + \dots + i_N(t_0) \end{aligned}$$

$$i = \frac{1}{L_{eq}} \int_{t_0}^t v dt + i(t_0)$$

where,

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_N} \quad (5.15)$$

- Example 1:

Find the equivalent inductance of the circuit in Figure 5.17.

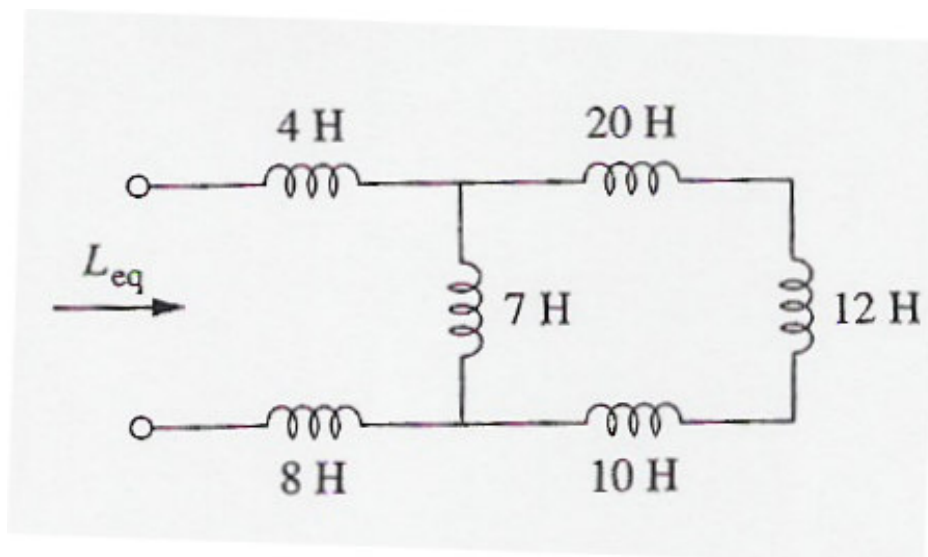


Figure 5.17

$$L_{eq} = 4H + 8H + (7H \parallel (20H + 12H + 10H))$$

$$\therefore L_{eq} = 18H$$

- Example 2:

For the circuit in Figure 5.18, $i(t) = 4(2 - e^{-10t})$ mA. If $i_2(0) = -1$ mA, find (a) $i_1(0)$, (b) $v(t)$, $v_1(t)$ and $v_2(t)$, (c) $i_1(t)$ and $i_2(t)$.

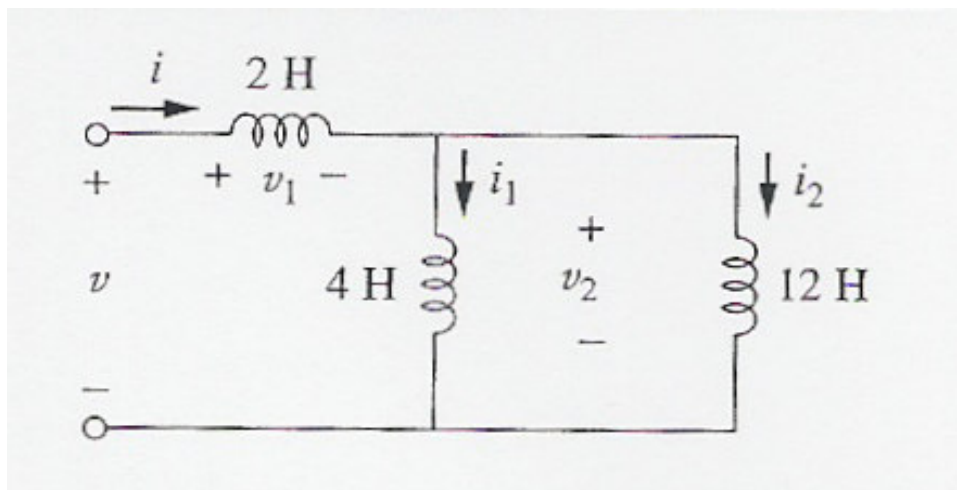


Figure 5.18

- (a) From $i(t) = 4(2 - e^{-10t})$ mA,
 $i(0) = 4(2 - 1) = 4$ mA

Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

- (b) The equivalent inductance is

$$L_{eq} = 2 + 4 \parallel 12 = 5 \text{ H}$$

Thus,

$$v(t) = L_{eq} \frac{di}{dt} = 5(4)(-1)(-10)e^{-10t} \text{ mV}$$

$$\therefore v(t) = 200e^{-10t} \text{ mV}$$

and

$$v_1(t) = L_{2H} \frac{di}{dt} = 2(-4)(-10)e^{-10t} \text{ mV}$$

$$\therefore v_1(t) = 80e^{-10t} \text{ mV}$$

Since $v = v_1 + v_2$,

$$v_2(t) = v(t) - v_1(t) = 120e^{-10t} \text{ mV}$$

(c) The current i_1 is obtained as,

$$\begin{aligned} i_1(t) &= \frac{1}{4} \int_0^t v_2 dt + i_1(0) \\ &= \frac{120}{4} \int_0^t e^{-10t} dt + 5 \text{ mA} \\ &= -3e^{-10t} \Big|_0^t + 5 \text{ mA} \\ &= -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t} \text{ mA} \end{aligned}$$

Similarly,

$$\begin{aligned} i_2(t) &= \frac{1}{12} \int_0^t v_2 dt + i_2(0) \\ &= \frac{120}{12} \int_0^t e^{-10t} dt - 1 \text{ mA} \\ &= -e^{-10t} \Big|_0^t - 1 \text{ mA} \\ &= -e^{-10t} + 1 - 1 = -e^{-10t} \text{ mA} \end{aligned}$$

5.6 Conclusion

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v - i :	$v = iR$	$v = \frac{1}{C} \int_{t_0}^t i dt + v(t_0)$	$v = L \frac{di}{dt}$
i - v :	$i = v/R$	$i = C \frac{dv}{dt}$	$i = \frac{1}{L} \int_{t_0}^t v dt + i(t_0)$
p or w :	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2} C v^2$	$w = \frac{1}{2} L i^2$
Series:	$R_{\text{eq}} = R_1 + R_2$	$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{\text{eq}} = L_1 + L_2$
Parallel:	$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{\text{eq}} = C_1 + C_2$	$L_{\text{eq}} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

[†]Passive sign convention is assumed.