CHAPTER 7: SECOND-ORDER CIRCUITS

7.1 Introduction

- This chapter considers circuits with two storage elements.
- Known as second-order circuits because their responses are described by differential equations that contain second derivatives.
- Example of second-order circuits are shown in figure 7.1 to 7.4.

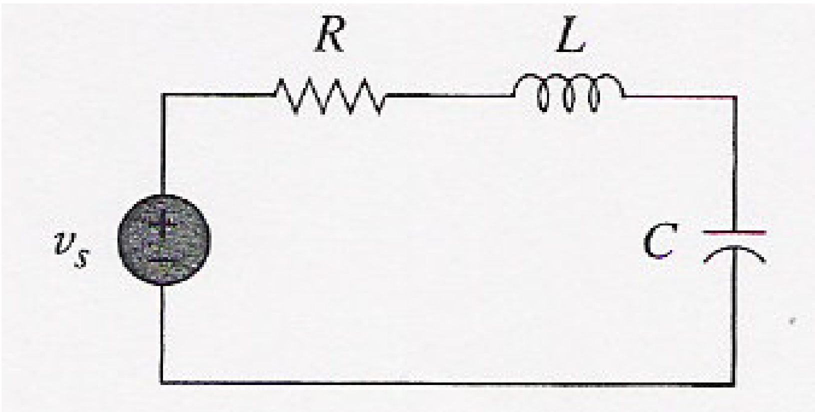

Figure 7.1

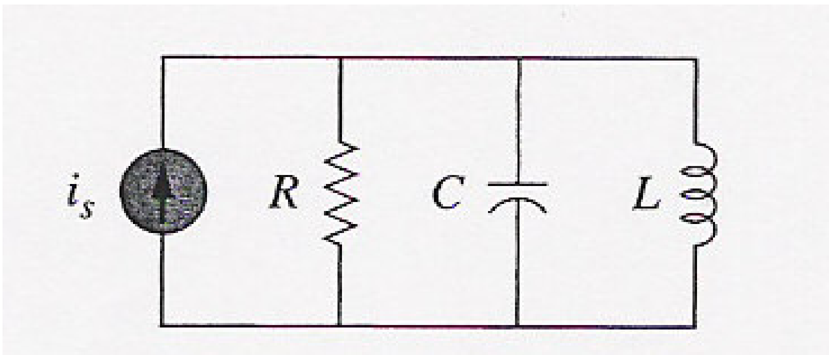

Figure 7.2

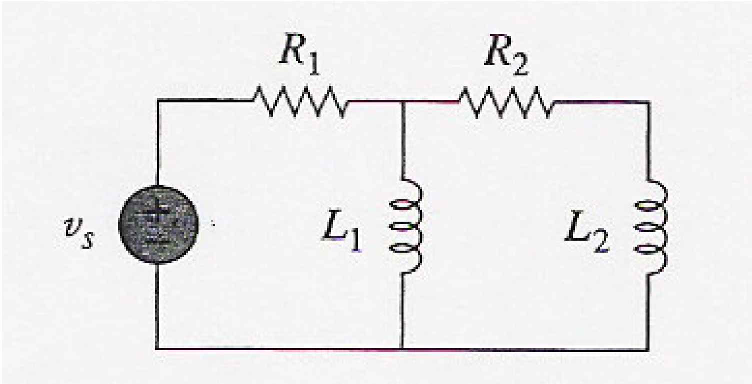

Figure 7.3

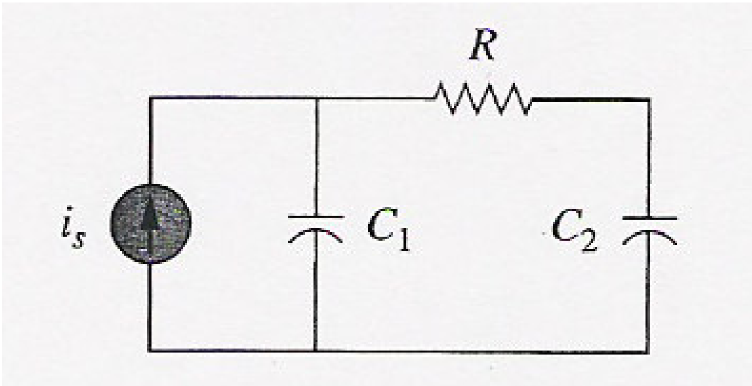

Figure 7.4

7.2 Finding Initial and Final Values

- Objective:
  Find $v(0), i(0), dv(0)/dt, di(0)/dt, i(\infty), v(\infty)$
- Two key points:
  (a) The direction of the current $i(t)$ and the polarity of voltage $v(t)$.

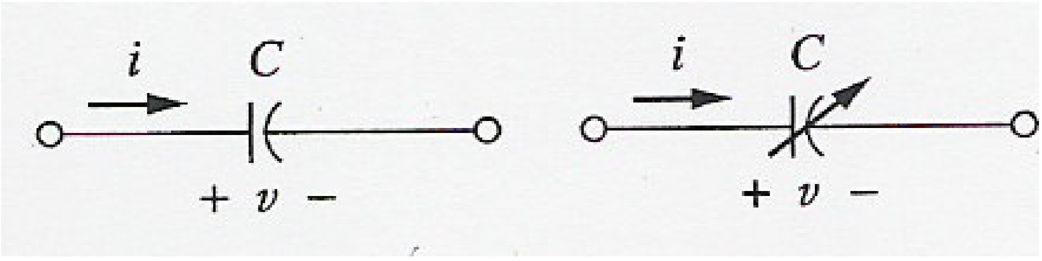

Figure 7.5

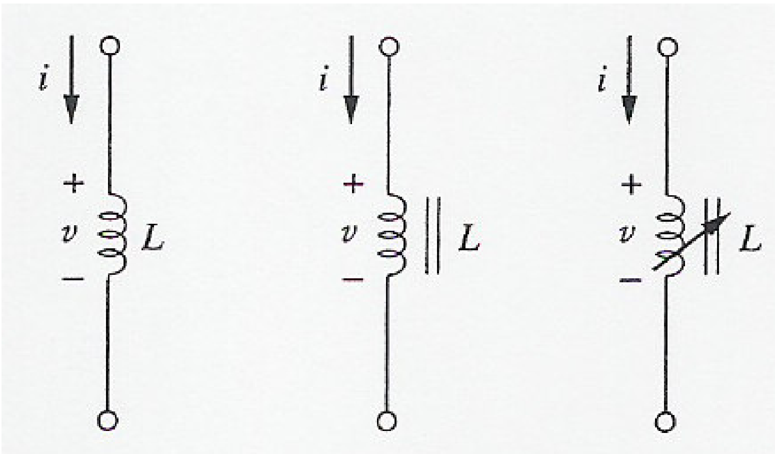

Figure 7.6

  (b) The capacitor voltage is always continuous:

$$v(0^+) = v(0^-)$$

and the inductor current is always continuous:

$$i(0^+) = i(0^-)$$

- Example:

The switch in Figure 7.7 has been closed for a long time. It is open at $t=0$. Find $i(0^+), v(0^+), di(0^+)/dt, dv(0^+)/dt, i(\infty), v(\infty)$

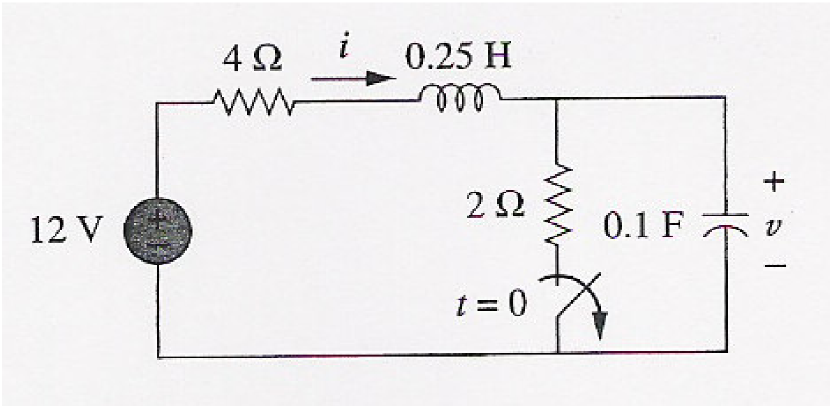

Figure 7.7

The switch is closed a long time before $t=0$, thus the circuit has reached dc steady state at $t=0$.
The inductor – acts like a short circuit.
The capacitor – acts like an open circuit.

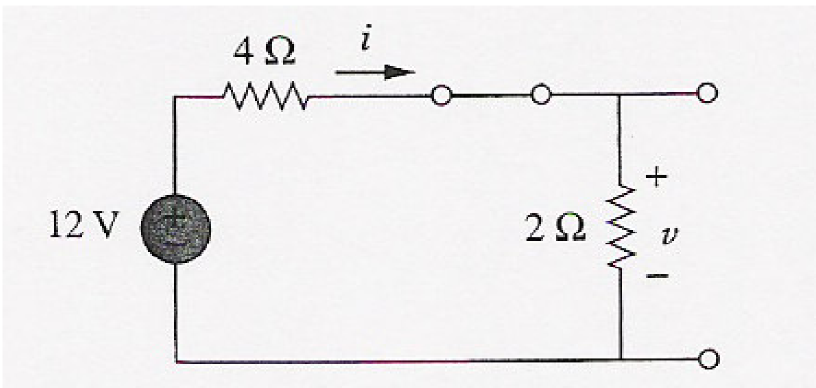

Figure 7.8

$$i(0^-) = \frac{12}{4+2} = 2\text{A}$$

$$v(0^-) = 2i(0^-) = 2(2) = 4\text{V}$$

As the inductor current and capacitor voltage cannot change abruptly,

$$i(0^+) = i(0^-) = 2\text{A}$$

$$v(0^+) = v(0^-) = 4\text{V}$$

At $t = 0^+$, the switch is open and the equvalent can be drawn as:

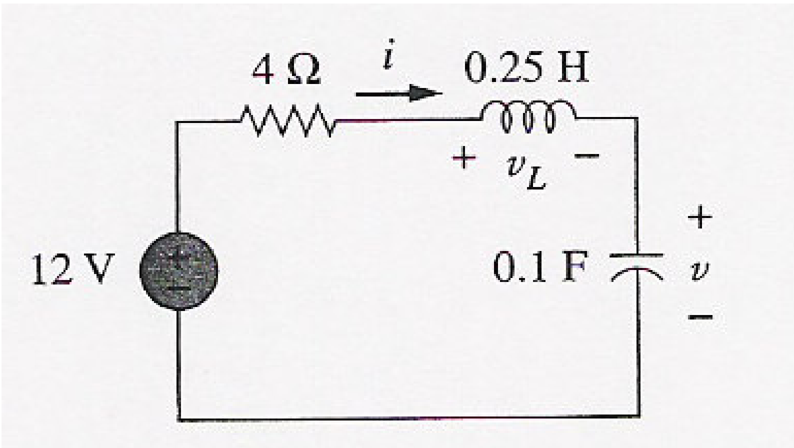

Figure 7.9

$$i_C(0^+) = i(0^+) = 2\text{A}$$

Since $C\dfrac{dv}{dt} = i_C, dv/dt = i_C/C$ and

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{V/s}$$

Similarly,

Since $L\,di/dt = v_L, di/dt = v_L/L$, applying KVL

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

$$v_L(0^+) = 12 - 8 - 4 = 0$$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\,\text{A/s}$$

For $t > 0$, the circuit undergoes transience.

But $t \to \infty$, the circuit reaches steady state again.

The inductor – acts like a short circuit.

The capacitor – acts like an open circuit.

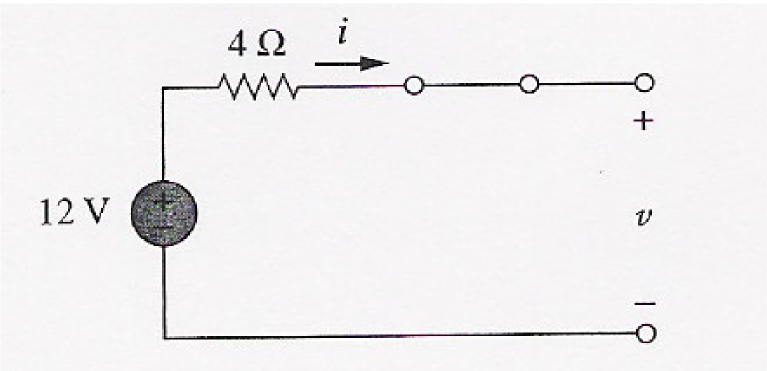

Figure 7.10

Thus,

$$i(\infty) = 0\,\text{A} \qquad v(\infty) = 12\,\text{V}$$

7.3 The Source-Free Series RLC Circuit

- Consider the source-free series RLC circuit in Figure 7.11.

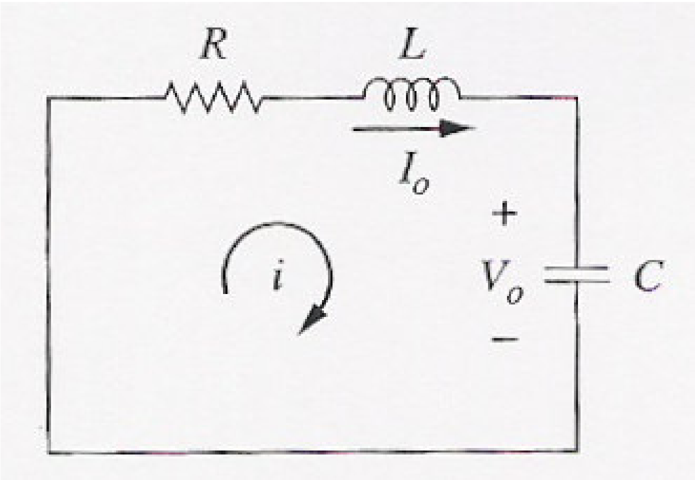

Figure 7.11

- The circuit is being excited by the energy initially stired in the capacitor and inductor.
- V_0 - the initial capacitor voltage

  I_0 - the initial inductor current
- Thus, at $t = 0$

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} idt = V_0$$

$$i(0) = I_0$$

- Applying KVL around the loop:

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} idt = 0$$

  Differentiate with respect to *t*:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

- the second-order differential equation

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

$$\frac{di(0)}{dt} = -\frac{1}{L}\left(RI_0 + V_0\right)$$

- Let $i = Ae^{st}$ - the exponential form for 1st order circuit
- Thus, we obtain

$$As^2 e^{st} + \frac{AR}{L} se^{st} + \frac{A}{LC} e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

or $$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

This quadratic equation is known as the *characteristic equation* since the root of the equation dictate the character of *i*.

- The 2 roots are:

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

or

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad (7.1)$$

where

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad (7.2)$$

- The roots s_1, s_2 are called naural frequencies, measured in nepers per second (Np/s).
  - they are associated with the natural response of the circuit.
- ω_0 is known as the resonant frequency or strictly as the undamped natural frequency, expressed in radians per second (rad/s).
- α is the neper frequency or the damping factor, expressed in nepers per second.
- 2 possible solutions for *i*:

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t}$$

- $\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0$ is a linear equation – any linear combination of the two distinct solutions i_1 and i_2 is also a solution for the equation.

  Thus,

  $$i(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

  where A_1 and A_2 are determined from the initia values $i(0)$ and $di(0)/dt$

- From Equation 7.1:

  (i) If $\alpha>\omega_0$ - overdamped case.

  (ii) If $\alpha=\omega_0$ - critically damped case.

  (iii) If $\alpha<\omega_0$ - underdamped case

- Overdamped case:

  - $\alpha>\omega_0$ implies $C>4L/R^2$.
  - both roots are negative and real.
  - The response,

  $$i(t)=A_1e^{s_1t}+A_2e^{s_2t} \qquad (7.3)$$

  which decays and approaches zero as *t* increases as shown in Figure 7.12

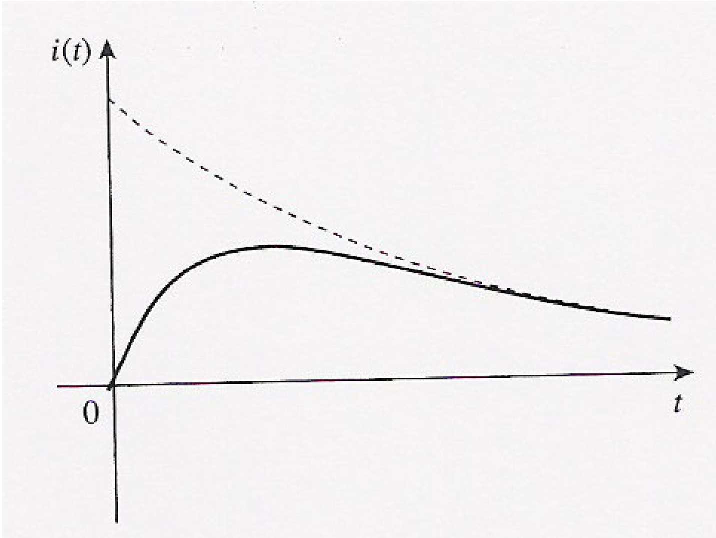

Figure 7.12

- Critically Damped Case:

  - $\alpha = \omega_0$ implies $C = 4L/R^2$

  - $s_1 = s_2 = -\alpha = -\dfrac{R}{2L}$

  - The response,

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

  where $A_3 = A_1 + A_2$

  - This cannot be the solution because the two initial conditions cannot be satisfied with the single constant A_3.

  - Let consider again:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

  - $\alpha = \omega_0 = R/2L$, thus,

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$$

- Let,

$$f = \frac{di}{dt} + \alpha i$$

- Thus,

$$\frac{df}{dt} + \alpha f = 0$$

which is the 1st order differential equation with solution $f = A_1 e^{-\alpha t}$

- So,

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1$$

which can be written as:

$$\frac{d}{dt}\left(e^{\alpha t} i\right) = A_1$$

- Intergrating both sides:

$$e^{\alpha t} i = A_1 t + A_2$$

or

$$i = (A_1 t + A_2)e^{\alpha t}$$

- Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term:

$$i(t) = (A_2 + A_1 t)e^{-\alpha t} \qquad (7.4)$$

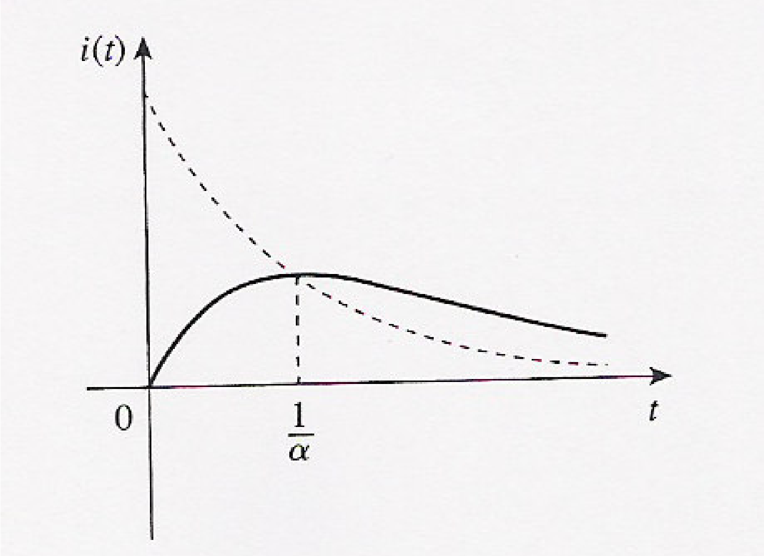

Figure 7.13

- Underdamped Case:
  - $\alpha < \omega_0$ implies $C < 4L/R^2$
  - The roots can be written as:

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*.

- Both ω_0 and ω_d are natural frequencies because they help determine the natural response.
- ω_0 is called the undamped natural frequency.
- ω_d is called the damped natural frequency.
- The natural response is

$$i(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$
$$= e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

- Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

- We get,

$$i(t) = e^{-\alpha t}\left[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)\right]$$

$$i(t) = e^{-\alpha t}\left[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\right]$$

- Replacing constant $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constant B_1 and B_2, we get

$$i(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right) \quad (7.5)$$

- With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature.
- The response has a time constant of $1/\alpha$ and a period of $T = 2\pi / \omega_d$

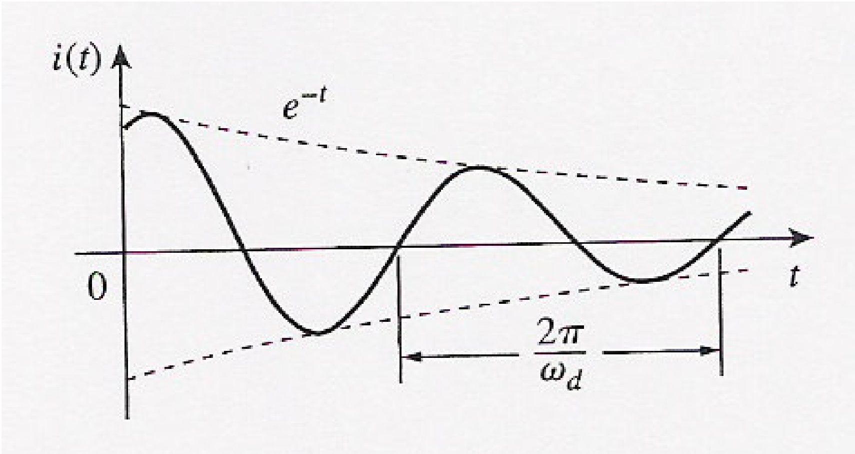

Figure 7.14

- Conclusions:

(i) - The behaviour of such network is captured by the idea of damping, which is the gradual loss of the initial stored energy.
- The damping effect is due to the presence of resistance R.
- The damping factor α determines the rate at which the response is damped.
- If $R = 0$, then $\alpha = 0$ and we have an LC circuit with $1/\sqrt{LC}$ as the undamped natural frequency. Since $\alpha < \omega_0$ in this

case, the response is not only undamped but also oscillatory.

- The circuit is said to be *lossless* because the dissipating or damping element (R) is absent.
- By adjusting the value of R, the response may be made undamped, overdamped, critically damped or underdamped.

(ii)
- Oscillatory response is possible due to the presence of the two types of storage elements.
- Having both L and C allows the flow of energy back and forth between the two.
- The damped oscillation exhibited by the underdamped response is known as ringing.
- It stems from the ability of the storage elements L and C to transfer energy back and forth between them.

(iii)
- It is difficult to differentiate between the overdamped and critically damped response.
- the critically damped response is borderline and decays the fastest.
- The overdamped has the longest settling time because it takes the longest time to dissipate the initial stored energy.

- If we desire the fastest response without oscillation or ringing, the critically damped circuit is the right choice.

- Example:
  In Figure 7.15, $R = 40\Omega, L = 4H, C = 1/4F$. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped or critically damped.

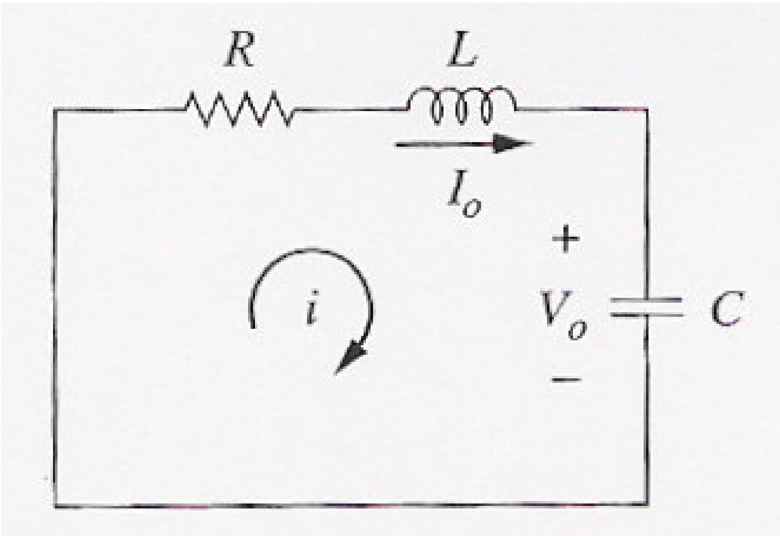

Figure 7.15

$$\alpha = \frac{R}{2L} = 5, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

$$s_1 = -0.101, \quad s_2 = -9.899$$

Since $\alpha > \omega_0$, the response is overdamped.

7.4 The Source-Free Parallel RLC Circuit

- Parallel RLC circuits find many practical applications – e.g. incommunications networks and filter designs.
- Consider the parallel RLC circuit shown in Figure 7.16:

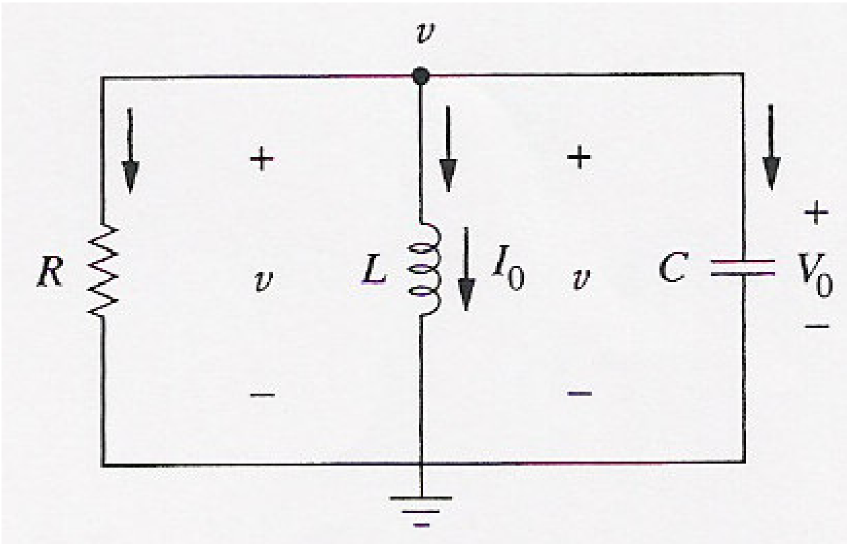

Figure 7.16

- Assume initial inductor current I_0 and initial capacitor voltage V_0.

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)dt$$

$$v(0) = V_0$$

- Since the three elements are in parallel, they have the same voltage v across them.
- According to passive sign conention, the current is entering each element

- the current through each element is leaving the top node.

- Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\, dt + C\frac{dv}{dt} = 0$$

- Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

- Replace the first derivative by s and the second derivative by s^2.
- Thus,

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

- The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad (7.6)$$

where

$$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad (7.7)$$

- There are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$.

Overdamped Case ($\alpha > \omega_0$)

- $\alpha > \omega_0$ when $L > 4R^2C$.
- The roots of the characteristic equation are real and negative
- The response is

$$v(t) = A_1 e^{s1t} + A_2 e^{s2t} \qquad (7.8)$$

Critically Damped Case ($\alpha = \omega_0$)

- For $\alpha = \omega$, $L = 4R^2C$.
- The roots are real and equal
- The response is

$$v(t) = (A_1 + A_2 t)\, e^{-\alpha t} \qquad (7.9)$$

Underdamped Case ($\alpha < \omega_0$)

- When $\alpha < \omega_0$, $L < 4R^2C$.
- The roots are complex and may be expressed as

$$S_{1,2} = -\alpha \pm j\omega_d$$

Where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

- The response is

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t \ + \ A_2 \sin \omega_d t) \quad (7.10)$$

- The constants A1 and A2 in each case can be determined from the initial conditions.
- We need *v*(*0*) and *dv*(*0*)/*dt*.
- The first term is known from:

$$v(0) = V_0$$

- For second term is known by combining

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)dt$$

$$v(0) = V_0$$

and

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v \ dt + C\frac{dv}{dt} = 0$$

as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$

- The voltage waveforms are similar to those shown in Section 7.3.
- Having found the capacitor voltage $v(t)$ for the parallel RLC circuit as shown above, we can readily obtain other circuit quantities such as individual element currents.
- For example, the resistor current is $i_R = v/R$ and the capacitor voltage is $v_C = C\, dv/dt$.
- Notice that we first found the inductor current $i(t)$ for the RLC series circuit, whereas we first found the capacitor voltage $v(t)$ for the parallel RLC circuit.

- Example:
  In the parallel circuit of Figure 7.17, find $v(t)$ for $t > 0$, assuming $v(0)$ = 5V, $i(0)$ = 0, L = 1H and C = 10mF. Consider these cases: $R = 1.923\Omega$, $R = 5\Omega$, and $R = 6.25\Omega$.

CASE 1 If R = 1.923 Ω

$$\alpha = \frac{1}{2RC} = \frac{1}{2\,x\,1.923\,x\,10\,x\,10\,x\,10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\,x\,10\,x\,10^{-3}}} = 10$$

Since $\alpha > \omega_0$, the response is overdamped.

The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t}$$

We now apply the initial conditions to get A_1 and A_2.

$$v(0) = 5 = A_1 + A_2$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC}$$

$$\frac{dv(0)}{dt} = -\frac{5+0}{1.923\,x\,10\,x\,10^{-3}} = 260$$

From $v(t) = A_1 e^{-2t} + A_2 e^{-50t}$,

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At t=0,

$$260 = -2A_1 - 50A_2$$

Thus,

$A_1 = 10.625$ and $A_2 = -5.625$

and

$$\mathrm{v(t)} = 10.625\mathrm{e}^{-2t} - 5.625\mathrm{e}^{-50t}$$

CASE 2 When R = 5Ω

$$\alpha = \frac{1}{2RC} = \frac{1}{2\,x\,5\,x\,10\,x\,10^{-3}} = 10$$

While ω_0 = 10 remains the same.

Since $\alpha = \omega_0 = 10$, the response is critically damped.

Hence, $s_1 = s_2 = -10$, and

$$\mathrm{v(t) = (A_1 + A_2 t)e^{-10t}}$$

To get A_1 and A_2, we apply the initial conditions

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5+0}{5\,x\,10\,x\,10^{-3}} = 100$$

From $\mathrm{v(t) = (A_1 + A_2 t)e^{-10t}}$,

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)\,e^{-10t}$$

At t=0

$$100 = -10A_1 + A_2$$

Thus,

$$A_1 = 5 \text{ and } A_2 = 150$$

and

$$\mathrm{v(t) = (5 + 150t)e^{-10t}\ \ V}$$

CASE 3 When R = 6.25 Ω

$$\alpha = \frac{1}{2RC} = \frac{1}{2\ x\ 6.25\ x\ 10\ x\ 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same.

As $\alpha < \omega_0$ in this case, the response is underdamped.

The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$$

We now obtain A_1 and A_2, as

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5+0}{6.25\ x\ 10\ x\ 10^{-3}} = 80$$

From $v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$,

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t$$

$$+ 6A_2 \cos 6t)\, e^{-8t}$$

At $t = 0$,

$$80 = -8A_1 + 6A_2$$

Thus,

$A_1 = 5$ and $A_2 = 20$.

and

$$v(t) = (5 \cos 6t + 20 \sin 6t)\, e^{-8t}$$

Note: by increasing the value of R, the degree of damping decreases and the responses differ.
The responses for those three cases:

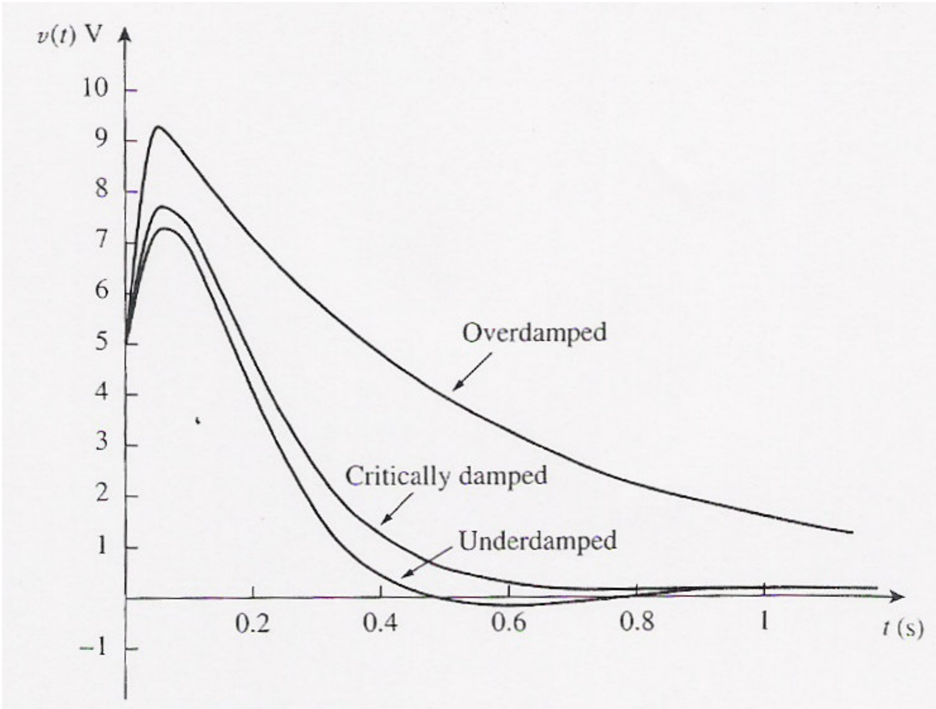

Figure 7.17

7.5 Step Response of a Series RLC Circuit

- Revision: the step response is obtained by the sudden application of a dc source.
- Consider the series RLC circuit shown in Figure 7.18.

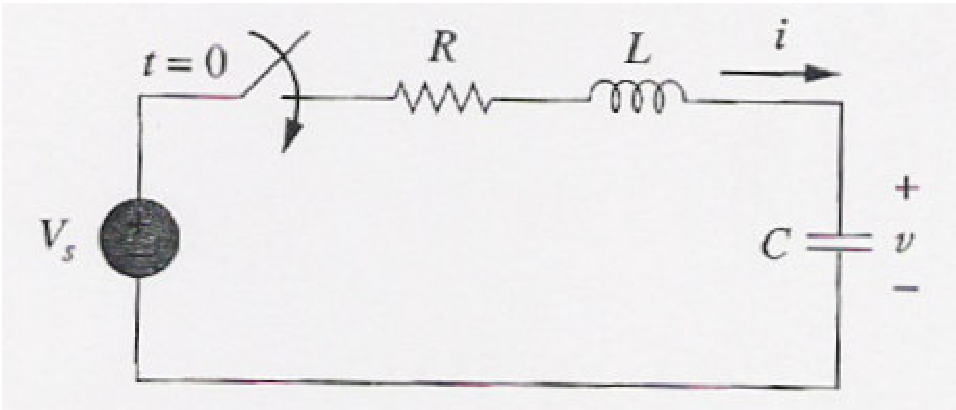

Figure 7.18

- Applying KVL around the loop for $t > 0$,

$$L\frac{di}{dt} + Ri + v = V_s$$

But

$$i = C\frac{dv}{dt}$$

Substituting for i and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

- The solution to the equation has two components: the transient response $v_t(t)$ and the steady-state response $v_{ss}(t)$;

$$v(t) = v_t(t) + v_{ss}(t)$$

- The transient response $v_t(t)$ is the component of the total response that dies out with time.
- The form of the transient response is the same as the form of the solution obtained in Section 7.3.
- Therefore, the transient response $v_t(t)$ for the overdamped, underdamped and critically damped cases are:

$$v_t(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (Overdamped)$$

$$v_t(t) = \left(A_1 + A_2 t\right) e^{-\alpha t} \quad (Critically\ damped)$$

$$v_t(t) = \left(A_1 \cos \omega_d t + A_2 \sin \omega_d t\right) e^{-\alpha t} \quad (Underdamped)$$

- The steady-state response is the final value of $v(t)$.
- In the circuit in Figure 7.18 the final value of the capacitor voltage is the same as the source voltage V_s.
- Hence,

$$v_{ss}(t) = v(\infty) = V_s$$

- Thus, the complete solutions for the overdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (Overdamped)$$
$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \quad (Critically\ damped)$$
$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad (Underdamped)$$

(7.11)

- The values of the constants A_1 and A_2 are obtained from the initial conditions: $v(0)$ and $dv(0)/dt$.
- Note: v and i are respectively, the voltage across the capacitor and the current through the inductor.
- Therefore, the Eq. 7.11 only applies for finding v.
- But once the capacitor voltage $v_C = v$ is known we can determine $i = C\, dv/dt$, which is the same current through the capacitor, inductor and resistor.
- Hence, the voltage across the resistor is $v_R = iR$, while the inductor voltage is $v_L = L\, di/dt$.
- Alternatively, the complete response for any variable *x(t)* can be found directly, because it has the general from

$$x(t) = x_{ss}(t) + x_t(t)$$

Where the $x_{ss} = x(\infty)$ is the final value and $x_t(t)$ is the transient response. The final value is found as in Section 7.2.

- Example

For the circuit in Figure 7.19, find *v(t)* and *i(t)* for t > 0. Consider these cases: R = 5 Ω.

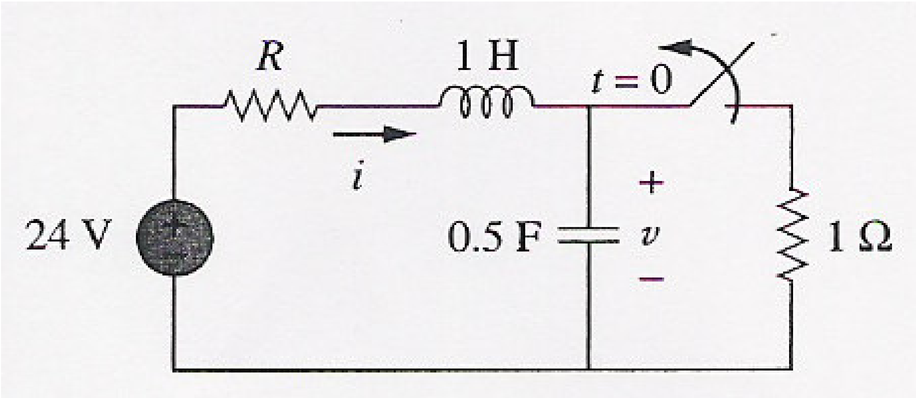

Figure 7.19

For t < 0, the switch is closed.

The capacitor behaves like an open circuit while the inductor acts like a short circuit.

The initial current through the inductor is

$$i(0) = \frac{24}{5+1} = 4A$$

And the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor; that is,

$$v(0) = 1i(0) = 4V$$

For t > 0, the switch is opened, so the 1-Ω resistor disconnected.

What remains is the series RLC circuit with the voltage source.

The characteristic roots are determined as follows.

$$\alpha = \frac{R}{2L} = \frac{5}{2\times 1} = 2.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$, we have the overdamped natural response.

The total response is therefore

$$v(t) = v_{ss} + \left(A_1 e^{-t} + A_2 e^{-4t}\right)$$

where v_{ss} is the steady-state response. It is the final value of the capacitor voltage.

In Figure 7.18 $v_f = 24$ V. Thus,

$$v(t) = 24 + \left(A_1 e^{-t} + A_2 e^{-4t}\right)$$

Find A_1 and A_2 using the initial conditions

$$v(0) = 4 = 24 + A_1 + A_2$$

or

$$-20 = A_1 + A_2$$

The current through the inductor cannot change abruptly and is the same current through the capacitor at $t = 0^+$ because the inductor and capacitor are now in series.

Hence,

$$i(0) = C\frac{dv(0)}{dt} = 4$$

$$\frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

From $v(t) = 24 + \left(A_1 e^{-t} + A_2 e^{-4t}\right)$,

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2$$

Thus,

A_1 = -64/3 and A_2 = 4/3.

and

$$v(t) = 24 + \frac{4}{3}\left(-16e^{-t} + e^{-4t}\right)V$$

since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current.

Hence,

$$i(t) = C\frac{dv}{dt}$$

Therefore,

$$i(t) = \frac{4}{3}\left(4e^{-t} - e^{-4t}\right)A$$

Note that *i(0)* = 4 A, as expected

7.6 Step Response of a Parallel RLC Circuit

- Consider the parallel RLC circuit shown in Figure 7.20.

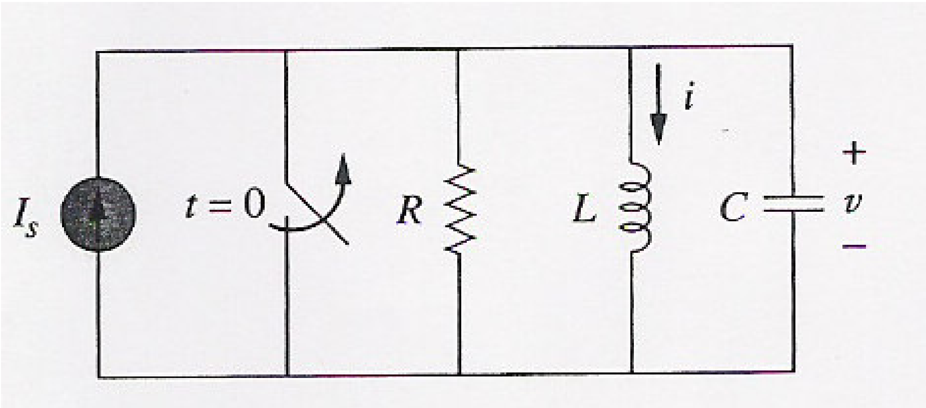

Figure 7.20

- Objective:
  Find *i* due to a sudden application of a dc current.
- Applying KCL at the top node for $t > 0$,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s$$

But

$$v = L\frac{di}{dt}$$

Substituting for *v* and dividing by *LC*,

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

- The complete solution consists of the transient response $i_t(t)$ and the steady-state response i_{ss};

$$i(t) = i_t(t) + i_{ss}(t)$$

- The steady-state response is the final value of *i*.
- In the circuit in Figure 7.20, the final value of the current through the inductor is the same as the source current I_s,
- Thus,

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

→ Overdamped

$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t}$$

→ Critically damped

$$i(t) = I_s + (A_1 \cos \omega_d t + A2 \sin \omega_d t)e^{-\alpha t}$$

→ Underdamped

- The constants A_1 and A_2 in each case can be determined from the initial conditions for *i* and *di/dt*.
- First, find the inductor current *i*.
- Once the inductor current $i_L = i$ is known, we can find $v = L\, di/dt$, which is the same voltage across inductor, capacitor and resistor.
- Hence, the current through the resistor is $i_R = v/R$, while the capacitor current is $i_C = C\, dv/dt$.
- Alternatively, the complete response for any variable $x(t)$ may be found directly, using

$$x(t) = x_{ss}(t) + x_t(t)$$

where x_{ss} and x_t are its final value and transient response, respectively.

- Example

In the circuit in Figure 7.21 find $i(t)$ and $i_R(t)$ for $t > 0$.

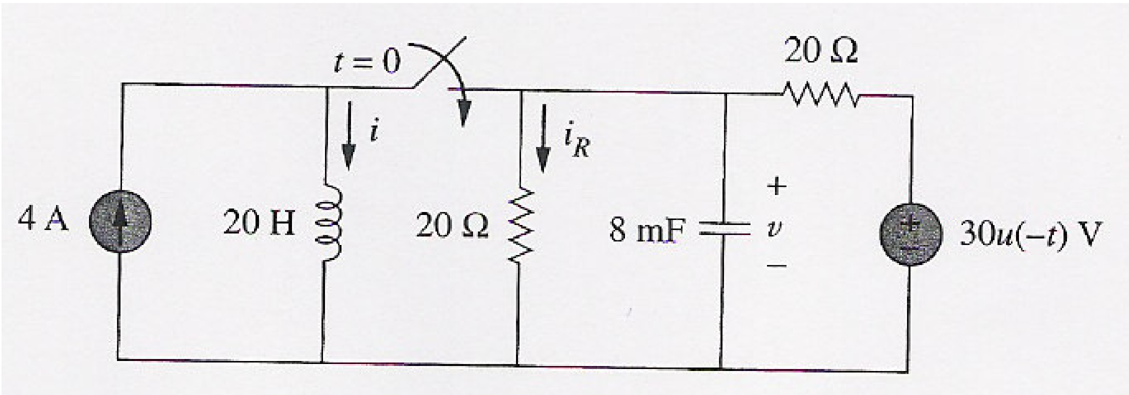

Figure 7.21

For t < 0, the switch is open and the circuit is partitioned into two independent subcircuits.

The 4-A current flows through the inductor, so that

$$i(0) = 4 \text{ A}$$

Since $30u(-t) = 30$ when $t < 0$ and 0 when $t > 0$, the voltage source is operative for $t < 0$ under consideration.

The capacitor acts like an open circuit and the voltage across it is the same as the voltage across the 20-Ω resistor connected in parallel with it.

By voltage division, the initial capacitor voltage is

$$v(0) = \frac{20}{20+20}(30) = 15V$$

For t > 0, the switch is closed and we have a parallel RLC circuit with a current source.

The voltage source is off or short-circuited.

The two 20-Ω resistors are now in parallel.

They are combined to give $R = 20 \| 20 = 10\Omega$.

The characteristic roots are determined as follows:

$$\alpha = \frac{1}{2RC} = \frac{1}{2\,x\,10\,x\,8\,x\,10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20\,x\,8\,x\,10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{39.0625 - 6.25}$$

$$= -6.25 \pm 5.7282$$

or

s_1 = - 11.978, s_2 = - 0.5218

Since $\alpha > \omega_0$, we have the overdamped case. Hence,

$$i(t) = I_s + A_1 e^{-11.978t} + A_2 e^{-0.5218t}$$

where $I_s = 4$ is the final value of $i(t)$.

Now use the initial conditions to determine A_1 and A_2.

At $t = 0$,

$i(0) = 4 = 4 + A_1 + A_2$

$A_2 = -A_1$

Taking the derivative of $i(t)$ in $i(t) = I_s + A_1e^{-11.978t} + A_2e^{-0.5218t}$

$$\frac{di}{dt} = -11.978A_1e^{-11.978t} - 0.5218A_2e^{-0.5218t}$$

so that at $t = 0$,

$$\frac{di(0)}{dt} = -11.978A_1 - 0.5218A_2$$

But

$$L\frac{di(0)}{dt} = v(0) = 15 \quad \Rightarrow \quad \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

Thus,

$0.75 = (11.978 - 0.5218)A_2$

$A_2 = 0.0655, \; A_1 = -0.0655$

The complete solution as

$i(t) = 4 + 0.0655\,(e^{-0.5218t} - e^{-11.978t})$ A

From $i(t)$, we obtain $v(t) = L\, di / dt$ and

$$i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785e^{-11.978t} - 0.0342e^{-0.5218t} \; A$$

7.7 General Second-Order Circuits

- Given a second-order circuit, we determine its step response $x(t)$ (which may be voltage or current) by taking the following four steps:
  1. First, determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$ as discussed in Section 7.2.
  2. Find the transient response $x_t(t)$ by applying KCL and KVL. Once a second-order differential equation is obtained, determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown constant as we did in the previous sections.
  3. Obtain the forced response as

     $$x_f(t) = x(\infty)$$

     where $x(\infty)$ is the final value of x, obtained in Step 1.
  4. The total response is now found as the sum of the transient response and steady-state response

     $$x(t) = x_t(t) + x_{ss}(t)$$

Finally determine the constant associated with the transient response by imposing the initial conditions $x(0)$ and $dx(0)/dt$, determined in step 1.

- Example:

Find the complete response v and then i for $t > 0$ in the circuit of Figure 7.22.

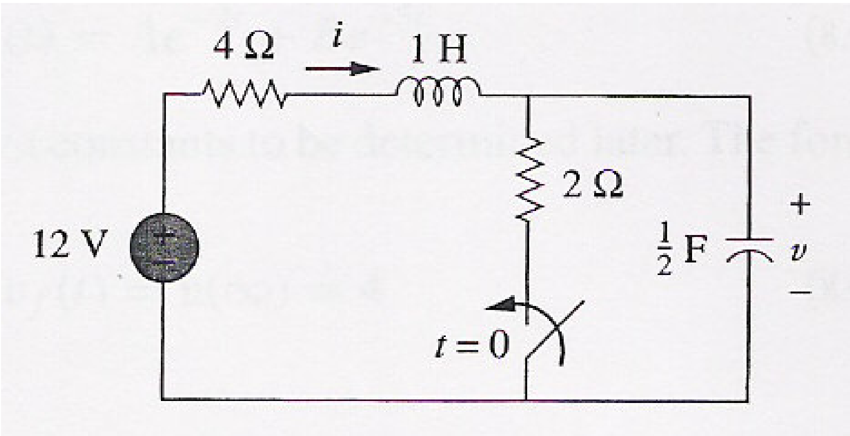

Figure 7.22

First find the initial and final values.

At $t > 0^{-}$, the circuit is at steady state. The switch is open, the equivalent circuit is shown in Figure 7.23.

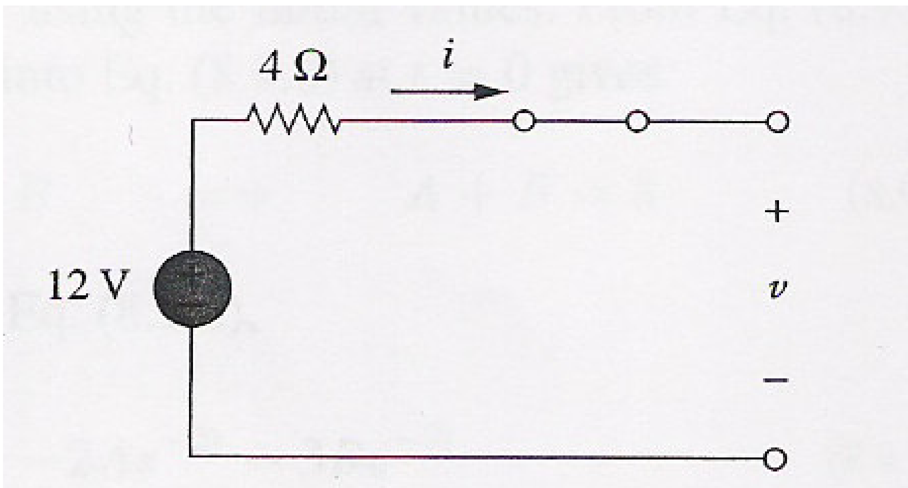

Figure 7.23

From the figure,

$$v(0^-) = 12V \qquad i(0^-) = 0$$

At $t > 0^+$, the switch is closed, the equivalent circuit is in Figure 7.24.

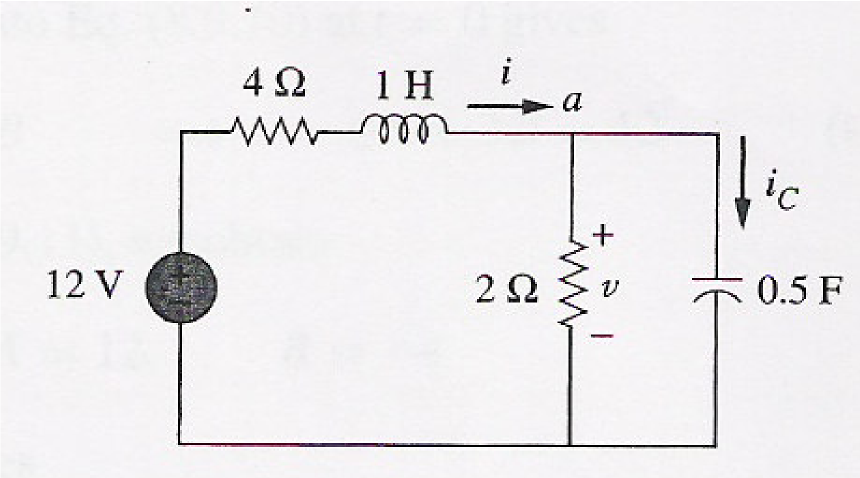

Figure 7.24

By the continuity of capacitor voltage and inductor current,

$$v(0^+) = v(0^-) = 12V \qquad i(0^+) = i(0^-) = 0$$

To get $dv > (0^+)/dt$,

use C *dv/dt* = i_c or *dv/dt* = i_C/C. Applying KCL at node *a* in Figure 7.24,

$$i(0^+) = i_c(0^+) + \frac{v(0^+)}{2}$$

$$0 = i_c(0^+) + \frac{12}{2} \qquad \Rightarrow \qquad i_c(0^+) = -6A$$

Hence

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12\,V/s$$

The final values are obtained when the inductor is replaced by a short circuit and the capacitor by an open circuit in Figure 7.24, giving

$$i(\infty) = \frac{12}{4+2} = 2A \qquad v(\infty) = 2i(\infty) = 4V$$

Next, obtain the natural response for $t > 0$.
By turning off the 12-V voltage source, we have the circuit in Figure 7.25.

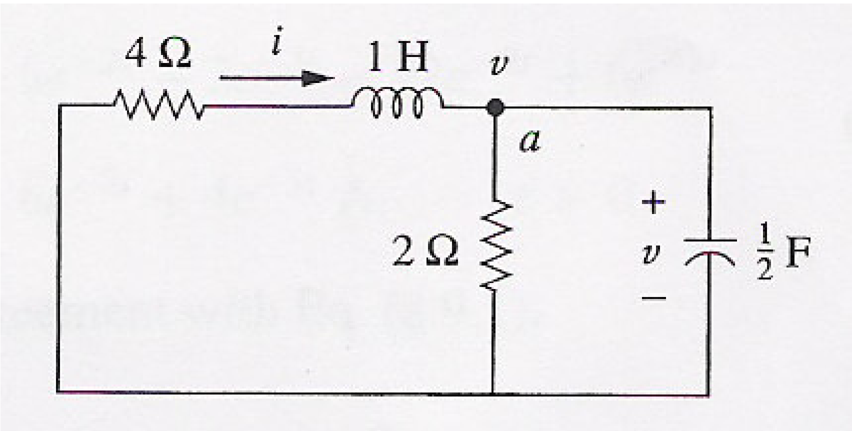

Figure 7.25

Applying KCL at node *a* in Figure 7.25 gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt}$$

Applying KVL to the left mesh results in

$$4i + 1\frac{di}{dt} + v = 0$$

Thus,

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

or

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

From this, we obtain the characteristic equation as

$$s^2 + 5s + 6 = 0$$

With roots s = -2 and s = -3. Thus, the natural response is

$$v_n(t) = Ae^{-2t} + Be^{-3t}$$

where A and B are unknown constants to be determined later.

The forced response is

$$v_f(t) = v(\infty) = 4$$

The complete response is

$$v(t) = v_n + v_f = 4 + Ae^{-2t} + Be^{-3t}$$

We now determine A and B using the initial values.

We know that $v(0) = 12$, thus at $t = 0$:

$$12 = 4 + A + B \qquad \Rightarrow \quad A + B = 8$$

Taking the derivative of v in

$$v(t) = v_n + v_f = 4 + Ae^{-2t} + Be^{-3t}$$

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t}$$

From $\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12\,V/s$, at $t = 0$:

$$-12 = -2A - 3B \qquad \Rightarrow \quad 2A + 3B = 12$$

Thus,

$$A = 12, \qquad B = -4$$

so that,

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}V, \qquad t > 0$$

From v, we can obtain other quantities of interest (refer to Figure 7.24):

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t}$$

$$= 2 - 6e^{-2t} + 4e^{-3t}A, \quad t > 0$$